TD Analyse S:2 SMPC Intégrale de Reimann

Faculté des Sciences Semlalia
SPMC S1 Analyse (2007-2008)

Série N^{0}4

Exercice 1 : En utilisant la définition de l'intégrale de Reimann, calculer les limites des suites suivantes :

$$i) u_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$$

$$ii) \; w_n = ((n+1)(n+2)....(n+k)...(n+n))^{\frac{1}{n}}$$

Exercice 2 : Soient les suites de termes généraux

$$u_n = \sum_{k=1}^{n} \frac{1}{n} \ln(1+\frac{k}{n}) \;,\; v_n = \frac{1}{n}(n!C_{2n}^{n})^{\frac{1}{n}} \;,\quad w_n = \sum_{k=1}^{n-1} \frac{n}{n^2+k^2}$$

i) Montrer que

$$\lim u_n = \int_1^2 \ln x dx$$

ii) En déduire que

$$\lim v_n = \exp(2\ln 2 - 1)$$

iii)

$$\lim w_n = \frac{\pi}{4}.$$

Exercice 3 : Par intégration par parties de

$$\int_0^1 \frac{dx}{x^2+1},$$

calculer

$$\int_0^1 \frac{dx}{(x^2+1)^2}$$

Exercice 4 : Calculer

$$i) \int_1^2 x^2 \ln x dx \quad , \quad ii) \int_1^2 x^2 \sin x dx \quad , \quad iii) \int_0^1 \sin x \exp(x) dx$$

Exercice 5 : En posant,

$$u = \sqrt{t-1}, \text{ calculer } \int_1^x \frac{dt}{1+\sqrt{t-1}} \quad \text{pour tout } x \geq 1.$$

Exercice 6 : Calculer

$$\int_0^1 (\cos x)^2 dx$$

Exercice 7 : Calculer

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx}.$$

Exercice 8 : Calculer

$$\int_0^1 \frac{x-1}{(x+1)^2\,(x^2+x+1)}dx$$

Exercice 9 : Calculer

$$\int_0^1 \frac{x^4+1}{(x+1)\,(x^2+x+1)}dx$$

Serie de calcule d'integrale.

Exercice 1

$S_n = \sum_{i=1}^{n} f(\lambda_i)(x_i - x_{i-1})$ pour chaque subdivision :

$a = x_0 < x_1 < x_2 \cdots < x_n = b$ $[a,b]$ somme de Riemann associée à la subdivision régulier à $(n+1)$ éléments.

$$S_{Riemann} = \frac{b-a}{n} \sum_{i=1}^{n} f(\lambda_i)$$

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \left(\frac{b-a}{n} \sum_{i=1}^{n} f(\lambda_i) \right) = \int_a^b f(x)\,dx.$$

i) - $U_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$

$$U_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2} = n \sum_{k=1}^{n} \frac{1}{n^2+k^2} = n \sum_{k=1}^{n} \frac{1}{n^2\left(1+\left(\frac{k}{n}\right)^2\right)} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\Rightarrow U_n = \frac{1}{n} \sum_{k=1}^{n} f(\lambda_i) \quad \text{avec} \quad f(\lambda_i) = \frac{1}{1+\left(\frac{i}{n}\right)^2}$$

On pose $f(x) = \frac{1}{1+x^2}$ pour $x \in [a,1]$; f est continue et positive $\forall x \in [0,1]$.

$$\text{donc } \lim_{n \to +\infty} U_n = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\operatorname{Arctg}(x)\right]_0^1$$

$$= \operatorname{Arctg}(1) - \operatorname{Arctg}(0) = \pi/4$$

ii) $W_n = \left((n+1)(n+2) - (n+k) - (n+n)\right)^{1/n}$

$$\operatorname{Log} W_n = \frac{1}{n}\left[\operatorname{Log}\left(\sum_{k=1}^{n} (n+k)\right)\right.$$

$$= \frac{1}{n} \sum_{k=1}^{n} \operatorname{Log}(n+k) = \frac{1}{n} \sum_{k=1}^{n} \operatorname{Log}\left(n\left(1+\frac{k}{n}\right)\right)$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left[\operatorname{Log} n + \operatorname{Log}\left(1+\frac{k}{n}\right)\right]$$

$$= \frac{1}{n} \sum_{k=1}^{n} \operatorname{Log} x + \frac{1}{n} \sum_{k=1}^{n} \operatorname{Log}\left(1+\frac{k}{n}\right)$$

①

$\longrightarrow \lim \log W_n = \log x + \underbrace{\int_1^2 \log(x)\,dx}$ $u' = 1$

on peut calculer $v = \log x$

par integration par partie

$\lim\limits_{n\to+\infty} e^{\ln W_n}$ $\lim\limits_{n\to+\infty} W_n = +\infty$

$\left(\int_1^2 \log(x)\,dx = cte\right)$

Car $\lim\limits_{n\to+\infty} e^{U_n} = \lim\limits_{n\to+\infty} \left(x + e^{}\right) = +\infty$

$\| \lim\limits_{n\to+\infty} W_n = +\infty$

Exercice 2

$\circledast$ $U_n = \sum\limits_{k=1}^{n} \frac{1}{n}\ln\left(1+\frac{k}{n}\right) = \frac{1}{n}\sum\limits_{k=1}^{n}\ln\left(1+\frac{k}{n}\right) = \frac{1}{n}\sum\limits_{k=1}^{n} f(\lambda_k)$

avec $I = [1,2]$ et $\lambda_k = \frac{k}{n}$ $f(x) = \ln(x)$.

$\Rightarrow \lim\limits_{n\to+\infty} U_n = \int_1^2 \ln(x)\,dx$ on calcule cet integrale par integration par partie $u' = 1$ et $v = \ln(x)$

$$\int_1^2 \ln(x)\,dx = \Big[x\ln(x)\Big]_1^2 - \int_1^2 x\cdot\frac{1}{x}\,dx = 2\ln(2) - 1$$

$\circledast$ $V_n = \frac{1}{n}\left(n!\, C_{2n}^{n}\right)^{\frac{1}{n}}$ avec $C_n^p = \frac{n!}{p!\,(n-p)!} \Rightarrow C_{2n}^{n} = \frac{(2n)!}{n!\,.\,n!}$

$\Rightarrow V_n = \frac{1}{n}\left(n! \cdot \frac{(2n)!}{(n!)^2}\right)^{1/n}$ $\frac{2n!}{n!} = \frac{\cancel{1\times2\times3\times\cdots(n)}\cdot(n+1)\times\cdots\times2n}{1.2.3.\cdots\cdots\times n}$

$V_n = \frac{1}{n}\left((n+1)(n+2)\cdots\times(n+n)\right)^{1/n}$

$\mathrm{Log}(V_n) = \frac{1}{n}\cdot\frac{1}{n}\mathrm{Log}\left(\sum\limits_{k=1}^{n} \ln(n+k)\right) = \frac{1}{n^2}\sum\limits_{k=1}^{n}\ln(n+k)$

$= \frac{1}{n^2}\sum\limits_{k=1}^{n}\mathrm{Log}(n) + \frac{1}{n^2}\sum\limits_{k=1}^{n}\mathrm{Log}\left(1+\frac{k}{n}\right)$

2

$$\operatorname{Log} V_n = \underbrace{\frac{n \operatorname{Log} n}{n^2}}_{n \to +\infty \ \to 0} + \frac{1}{n^2} \sum_{k=1}^{n} \operatorname{Log}\left(1 + \frac{k}{n}\right)$$

$$\operatorname{Log} V_n = -\log n + \frac{1}{n} \log \sum_{k=1}^{n} \log(n+k)$$

$$= -\log(n) + \frac{1}{n} \sum_{k=1}^{n} \left[\log(n) + \log\left(1 + \frac{k}{n}\right)\right]$$

$$= -\log(n) + \frac{1}{n} \sum_{k=1}^{n} \log(n) + \frac{1}{n} \sum_{k=1}^{n} \log\left(1 + \frac{k}{n}\right)$$

$$= -\log(n) + \frac{n}{n} \log(n) + \frac{1}{n} \sum_{k=1}^{n} \log\left(1 + \frac{k}{n}\right)$$

$$\Rightarrow \lim_{n \to +\infty} \operatorname{Log}(V_n) = \int_1^2 \log x \, dx = 2\log 2 - 1$$

par intégration par partie on pose $u' = 1$ et $v = \log(x)$

$$\Rightarrow \lim_{n \to +\infty} V_n = e^{2\log 2 - 1} = \frac{4}{e}.$$

$$* \; W_n = \sum_{k=1}^{n-1} \frac{n}{n^2 + k^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}}{1 + \left(\frac{k}{n}\right)^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{1 + \left(\frac{k}{n}\right)^2}$$

$$W_n = \frac{1}{n} \sum_{k=1}^{n-1} f(\lambda_k) \quad \text{avec } \lambda_k = \frac{k}{n} \quad \text{et } f(x) = \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{1+x^2} \quad \text{et } I = [0, 1]$$

$$\Rightarrow \lim_{n \to +\infty} W_n = \int_0^1 \frac{1}{1+x^2} = \left[\operatorname{Arctg}(x)\right]_0^1 = \operatorname{Arctg}(1) - \operatorname{Arctg}(0)$$

$$= \pi/4 - 0 = \pi/4$$

Exercice 3 Integration par partie

soit l'integrale $I = \int_0^1 \frac{dx}{1+x^2}$ on pose $U = \frac{1}{1+x^2}$; $V' = 1$

$U' = 1 \Rightarrow U = x$ et $V = \frac{1}{1+x^2} \longrightarrow V' = \frac{1}{(1+x^2)^2}$

soit $J = \int_0^1 \frac{dx}{(x^2+1)^2}$

$$I = \left[\frac{x}{1+x^2}\right]_0^1 + 2\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$$

$$= \frac{1}{2} + 2\int_0^1 \frac{(x^2+1)-1}{(1+x^2)^2}\,dx = \frac{1}{2} + 2\int_0^1 \frac{dx}{1+x^2} - 2\int_0^1 \frac{dx}{(1+x^2)^2}$$

$$\Rightarrow I = \frac{1}{2} + 2I - 2J \Rightarrow 2J = \frac{1}{2} + I = \frac{1}{2} + \operatorname{Arctg}(1)$$

$$\Rightarrow 2J = \frac{1}{2} + \pi/4 \Rightarrow \text{donc } J = \frac{2+\pi}{8}$$

$$* \ J = \int_0^1 \frac{dx}{(x^2+1)^2} = \frac{2+\pi}{8}$$

Exercice 4:

i) $I = \int_1^2 x^2 \ln(x)\,dx$ integration par partie

$u = \ln(x) \longrightarrow u' = \frac{1}{x}$

$V' = x^2 \longrightarrow V = \frac{x^3}{3}$

$$I = \left[\frac{x^3}{3}\ln(x)\right]_1^2 - \int_1^2 \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{8}{3}\ln(2) - \left[\frac{x^4}{3.4}\right]_1^2$$

$$I = \frac{8}{3}\ln(2) - \left[\frac{2^4}{3.4} - \frac{1}{3.4}\right]$$

ii) $K = \int_1^2 x^2 \sin x \, dx$ on pose $u = x^2 \rightarrow u' = 2x$

$v' = \sin x \rightarrow v = -\cos x$

$K = \left[-x^2 \cos x\right]_1^2 + 2\int_1^2 \underbrace{x}_{f} \underbrace{\cos x}_{g'} \, dx$ on fait un 2ème intégration par partie

$\int_1^2 x \cos x \, dx = \left[x \sin x\right]_1^2 + \int_1^2 \sin x \, dx$

iii) $L = \int_0^1 \sin x \, e^x \, dx$

$$\begin{cases} u' = e^x \rightarrow u = e^x \\ v = \sin x \rightarrow v' = \cos x \end{cases}$$

$\Rightarrow L = \left[e^x \sin x\right]_0^1 - \int_0^1 \underbrace{\cos x}_{f} \underbrace{e^x}_{g'} \, dx = \left[e^x \sin x\right]_0^1 + \left[\cos x e^x\right]_0^1 - \int_0^1 \sin x e^x dx$

$= \left[e^x \sin x\right]_0^1 + \left[\cos x e^x\right]_0^1 - L$

$\Rightarrow 2L = \left[e^x \sin x\right]_0^1 + \left[\cos x e^x\right]_0^1$

$L = \frac{1}{2}\left(\left[e^x \sin x\right]_0^1 + \left[\cos x e^x\right]_0^1\right)$

Exercice 5:

$I = \int_1^x \frac{dt}{1+\sqrt{t-1}}$ on pose $u = \sqrt{t-1} \Rightarrow u^2 = t-1$

$\Rightarrow dt = 2u \, du$

$t = 1 \rightarrow u = 0$ et $t = x \rightarrow u = \sqrt{x-1}$

$I = 2\int_0^{\sqrt{x-1}} \frac{[(u+1)-1] \, du}{1+u} = 2\int_0^{\sqrt{x-1}} du - 2\int_0^{\sqrt{x-1}} \frac{du}{1+u}$

$= 2\sqrt{x-1} - 2\left[\ln(1+u)\right]_0^{\sqrt{x-1}}$

$= 2\sqrt{x-1} - 2\left(\ln\left(\sqrt{x-1}+1\right) - 0\right)$

5

Exercice 6

$$I = \int_0^1 \cos^2 x \, dx \qquad \text{on a } \cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$= \int_0^1 \frac{\cos 2x + 1}{2} dx$$

$$= \int_0^1 \frac{1}{2} dx + \frac{1}{2}\int_0^1 \cos 2x \, dx = \frac{1}{2} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{4}\sin 2$$

Exercice 7

$$I = \int_0^{\pi/2} \frac{dx}{1+\sin x} \qquad \text{on pose } t = \operatorname{tg} x \rightarrow \sin x = \frac{2t}{1+t^2}$$

$$x = 2\operatorname{Arctg} t \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$x = 0 \rightarrow t = 0 \qquad dx = \frac{2dt}{1+t}$$

$$x = \pi/2 \rightarrow t = \operatorname{tg} \pi/4 = 1$$

$$\Rightarrow I = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = \int_0^1 \frac{2dt}{1+t^2+2t} = 2\int_0^1 \frac{dt}{(1+t)^2}$$

$$= 2\left[-\frac{1}{1+t}\right]_0^1 = 2\left[-\frac{1}{2}+1\right] = -1$$

Exercice 8

$$\text{soit } f(x) = \frac{x-1}{(x+1)^2 \underbrace{(x^2+x+1)}_{\Delta < 0}}$$

$$= \frac{a}{(x+1)} + \frac{b}{(x+1)^2} + \frac{cx+d}{x^2+x+1}$$

$$b = (x+1)^2 f(x)\Big|_{x=-1} = \frac{x-1}{x^2+x+1}\Big|_{x=-1} = -2.$$

$$(x^2+x+1) f(x)\Big|_{x=0} = a+b+d=-1 \Rightarrow a=1-d$$

$$(x^2+x+1) f(x)\Big|_{x=1} = \frac{3}{2}a+\frac{3}{4}b+c+d=0$$

$$\lim_{x\to+\infty} (x+1) f(x) = a+0+c=0 \Rightarrow a=-c$$

$$\begin{cases} b=-2 \\ a=-c \\ a=1-d \\ \frac{3}{2}a+\frac{3}{4}b+c+d=0 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=-2 \\ c=1 \\ d=2 \end{cases}$$

donc $f(x) = -\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{x+2}{x^2+x+1}$

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x-1}{(x+1)^2(x^2+x+1)}dx = -\int_0^1 \frac{dx}{1+x} - 2\int_0^1 \frac{dx}{(1+x)^2} + \frac{1}{2}\int_0^1 \frac{2x+1+3}{x^2+x+1}dx$$

$$= -\left[\ln|1+x|\right]_0^1 + 2\left[\frac{1}{1+x}\right]_0^1 + \frac{1}{2}\left[\ln|x^2+x+1|\right]_0^1$$

$$+\frac{3}{2}\int_0^1 \frac{dx}{\left(\frac{2x+1}{2}\right)^2+\frac{3}{4}} \quad \left(x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)$$

On a $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a}\int \frac{\frac{1}{a}dx}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a}\operatorname{Arctg}\left(\frac{x}{a}\right)$

donc $\displaystyle\int_0^1 \frac{dx}{\left(\frac{2x+1}{2}\right)^2+\frac{3}{4}} = \int_{1/2}^{3/2} \frac{dt}{t^2+\frac{3}{4}}$ on pose $t = x+\frac{1}{2} \Rightarrow dt=dx$

$x\to 0 \Rightarrow t\to 1/2$

$x\to 1 \Rightarrow t\to 3/2$

$$= \int_{1/2}^{3/2} \frac{dt}{\left(\frac{2t}{\sqrt{3}}\right)^2+1} = L$$

on fait un 2ème changement : on pose $u=\frac{2t}{\sqrt{3}} \to du=\frac{2}{\sqrt{3}}dt$

$$\Rightarrow L = \frac{\sqrt{3}}{2}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{3/4(u^2+1)} = \frac{2}{\sqrt{3}}\operatorname{Arctg}\frac{2t}{3} = \frac{2}{\sqrt{3}}\operatorname{Arctg}\left(\frac{2\left(x+\frac{1}{2}\right)}{3}\right)$$

7

Exercice 9 :

$$I = \int_0^1 \frac{x^4+1}{(x+1)\underbrace{(x^2+x+1)}_{\Delta<0}}\,dx$$

Division euclidienne :

$$\begin{array}{r|l} x^4 + 1 & x^3+2x^2+2x+1 \\ \hline -x^4-2x^3-2x^2-x & x-2 \\ -2x^3-2x^2-x+1 & \\ 2x^3+4x^2+4x+2 & \\ \hline 2x^2+3x+3 & \end{array}$$

$$I = \int_0^1 \frac{(x-2)(x+1)(x^2+x+1) + 2x^2+3x+3}{(x+1)(x^2+x+1)}$$

$$= \int_0^1 \left(x-2+\frac{2x^2+3x+3}{(x+1)(x^2+x+1)}\right)dx = \int_0^1 (x-2)\,dx + \int_0^1 g(x)\,dx$$

$$g(x) = \frac{a}{x+1} + \frac{bx+c}{x^2+x+1} \quad \text{avec : } (x+1)g(x)\Big|_{x=-1} = a = \frac{2x^2+3x+3}{x^2+x+1}\Big|_{x=-1} = \frac{2-3+3}{1} = 2$$

$$(x^2+x+1)g(x)\Big|_{x=0} = 3a+c = 3$$

$$(x^2+x+1)g(x)\Big|_{x=1} = \frac{1}{2}(3)a+b+c = 4$$

$$\Rightarrow b=0\ ;\ a=2 \text{ et } c=1$$

$$g(x) = \frac{2}{x+1} + \frac{1}{x^2+x+1}$$

$$\text{d'où } I = \int_0^1 (x-2)\,dx + \int_0^1 \frac{2}{x+1}\,dx + \int_0^1 \frac{dx}{x^2+x+1}$$

$$= \left[\frac{x^2}{2}-2x\right]_0^1 + 2\left[\ln|x+1|\right]_0^1 + \int_0^1 \frac{dx}{x^2+x+1}$$

$$\text{On a } \int_0^1 \frac{dx}{x^2+x+1} = \int_0^1 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int_{3/4}^{3/2} \frac{dt}{t^2+3/4} = F$$

$$\left(\begin{array}{l} t = x+\frac{1}{2} \Rightarrow dt = dx \\ t \to 1/2 \text{ quand } x \to 1 \\ t \to 1/2 \text{ quand } x \to 0 \end{array}\right.$$

$$F = \int_{1/2}^{3/2} \frac{dt}{t^2+\frac{3}{4}} = \int_{1/2}^{3/2} \frac{dt}{\frac{3}{4}\left[\left(\frac{2t}{\sqrt{3}}\right)^2+1\right]}$$

$$\text{on pose } u = \frac{2t}{\sqrt{3}} \to du = \frac{2}{\sqrt{3}}\,dt$$

$$\Rightarrow F = \frac{2}{\sqrt{3}}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{du}{u^2+1} = \frac{2}{\sqrt{3}}\left[\mathrm{Arctg}\,u\right]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} = \frac{2}{\sqrt{3}}\mathrm{Arctg}\left(\frac{2t}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\mathrm{Arctg}\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)$$

8